Fuzzy Belief Revision

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Abstract

Fuzzy sets, having been the long-standing mainstay of modeling and manipulating imperfect information, are an obvious candidate for representing uncertain beliefs.

Unfortunately, unadorned fuzzy sets are too limited to capture complex or potentially inconsistent beliefs, because all too often they reduce to absurdities ("nothing is possible") or trivialities ("everything is possible").

However, we show that by combining the syntax of propositional logic with the semantics of fuzzy sets a rich framework for expressing and manipulating uncertain beliefs can be created, admitting Gärdenfors-style expansion, revision, and contraction operators and being moreover amenable to easy integration with conventional "crisp" information processing.

The model presented here addresses many of the shortcomings of traditional approaches for building fuzzy data models, which will hopefully lead to a wider adoptance of fuzzy technologies for the creation of information systems.

Keywords: fuzzy belief revision, fuzzy information systems, soft computing, fuzzy object-oriented data model

1 INTRODUCTION

Fuzzy information systems combine two basic ideas: the creation of a "mini-world" needed for building database and information systems, with the realization that this mini-world should encompass the imprecision and vagueness that exists in the real world being modeled. The so-called *soft computing* approach strikes for better, more efficient systems by incorporating this imprecision into the data models used for building systems.

A theoretical foundation for modeling imperfect information is the well-known fuzzy set theory by Zadeh [Zad65]. However, it is not obvious how to incorporate these fuzzy sets into data models used for building information systems. Although there has been extensive research in the area of fuzzy data models, especially for the relational and object-oriented model [BK95, Pet96, MS97, Cal97, YG99, LXHY99], none of the proposed fuzzy models had any noticible impact for building actual fuzzy applications.

A reason for this might be that fuzzy information systems have a different set of requirements. While classical, precise information systems usually capture the state of a changing world, fuzzy systems are especially helpful in identifying possible worlds by managing changing, imperfect knowledge.

Here, the integration of imperfect information, possibly from different sources, becomes important. For such applications, however, simple fuzzy sets are no longer sufficient, since they cannot distinguish a single imperfect information from a complex state obtained by combining them. This makes it impossible to provide semantically rich operators for modifying imperfect data.

In this paper, we present a model that combines syntactical features from propositional logic with the semantics of fuzzy sets. In this model, atoms are imperfect predicates that have an underlying interpretation in form of a fuzzy set. More complex fuzzy clauses and formulas can be constructed from these fuzzy atoms through a fuzzy conjunctive normal form. We show how the AGM postulates for expansion and revision can be adapted for this fuzzy model, and how they result in efficient operators that can be used for building fuzzy systems.

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2 MODELING IMPERFECT INFORMATION

Existing fuzzy data models usually incorporate fuzziness by (at least) adding a new primitive data type "fuzzy set", e.g. for object attributes. While this seems like an obvious extension for introducing fuzziness into an object-oriented data model, we identified several shortcomings with this idea.

The core problem is that this approach overloads a single fuzzy set with both syntax and semantics. Imperfect information first has a *structure* that results from processing single pieces of imperfect information — e.g., by combining imperfect requirements, collecting vague information from many heterogenous information sources, or processing fuzzy results from different heuristics or sensors. Secondly, both a single piece of imperfect information and a result obtained by combining them has a *semantic* interpretation that provides the relevant meaning for the fuzzy set within a system. By mixing both syntax and semantics together into a single fuzzy set, neither can be clearly identified. As a side-effect, it becomes extremely difficult to provide meaningful operators to modify imperfect data. For example, consider the case where conflicting information (like many witnesses describing a person's height conflictingly from "very short" to "very tall") results in a useless fuzzy interpretation (a membership function that is constant one or constant zero): when working only with the fuzzy set representation it is impossible to precisely identify the conflicting information and selectively remove data in order to maintain a consistent knowledge base, which is an important rationale for database and information system practitioners. Dubois and Prade [DP94] already pointed out that purely semantical representations, like the one offered by possibilistic belief revision, are poorer than syntactic ones.

Consequently, what we need is a representation model for imperfect information that allows to selectively identify single pieces of information, yet offers precise semantics in terms of fuzzy sets. Classical propositional logic offers the first part: the construction of complex propositions from primitives, the *atoms*, and their logical combination to *clauses* and *formulas*. By substituting precise atoms with fuzzy atoms, we can construct fuzzy propositions where each logical constituent has a semantic interpretation in form of a fuzzy set. Modifications then become possible on the syntactical level, e.g. by adding a new imprecise fact or vague requirement with a logical "and" operation, which leads to a new semantic interpretation in form of a fuzzy set.

For example, adding a new requirement b to an existing imperfect requirement a results in the logical structure $a \wedge b$ as well as the fuzzy interpretation $\mu_{a\wedge b}$. This makes it possible to identify conflicts and inconsistencies on both the structural and fuzzy set level — if, for example, a new requirement is incompatible with $a \wedge b$ but not a or b alone, one of these syntactical components can be removed and the fuzzy interpretation be re-computed. The following commutative diagram illustrates this principle; here $\mathcal{F}$ denotes a fuzzy conjunctive normal form (which we will define below), I its interpretation as a fuzzy set and OP a possible operation (which we will define later):

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{\text{OP}} & \mathcal{F}' \\ \downarrow I & & \downarrow I \\ \mu_{\mathcal{F}} & \xrightarrow{\text{OP}'} & \mu_{\mathcal{F}'} \end{array}$$

With this separation it was possible to incorporate concepts from belief revision into fuzzy data models, which we will present in section 3.

The representation model introduced here is a reworked and generalized version of the model proposed by Boss [Bos96] for the domain of fuzzy design databases.

2.1 THE FUZZY REPRESENTATION MODEL

We first present the basic definitions for fuzzy sets and their operators [Zad65, KF88]:

Definition 2.1 (Fuzzy Sets and Operators) *A fuzzy set μ of Ω is a mapping of the reference set Ω to the [0,1]-interval:*

$$\mu : \Omega \rightarrow [0, 1]$$

We denote the set of all fuzzy sets over Ω with $F(\Omega)$.

The intersection of two fuzzy sets μ_1 and μ_2, written $\mu_1 \cap \mu_2$, is defined by:

$$(\mu_1 \cap \mu_2)(\omega) := \min\{\mu_1(\omega), \mu_2(\omega)\}, \omega \in \Omega$$

The union of two fuzzy sets μ_1 and μ_2, written $\mu_1 \cup \mu_2$, is defined by:

$$(\mu_1 \cup \mu_2)(\omega) := \max\{\mu_1(\omega), \mu_2(\omega)\}, \omega \in \Omega$$

Finally, the complement of a fuzzy set μ, written $\overline{\mu}$, is defined by:

$$\overline{\mu}(\omega) := 1 - \mu(\omega), \omega \in \Omega$$

For the purpose of this paper we only consider possibilistic interpretation of fuzzy sets. In this interpretation, a fuzzy set μ models an imprecise concept (predicate) C_μ, and a particular value $\mu(\omega)$ describes the consistency of ω with C_μ:

$\mu(\omega) = 0$	then ω is definitely inconsistent with C_μ,
$\mu(\omega) = 1$	means, ω is consistent with C_μ without restriction of any kind,
$\mu(\omega) \in (0, 1)$	gives the degree for the consistency of ω with C_μ.

Note that the case $\mu(\omega) = 1$ does *not* mean that C_μ is necessarily true of ω — it only means there is nothing that speaks against it.

We can now define our fuzzy representation model. As outlined above, we combine the structural features from propositional logic with the semantics of fuzzy sets.

The basic unit in this model is a *fuzzy atom,* which formalizes the notion of imprecise concepts introduced above:

Definition 2.2 (Fuzzy Concept Vocabulary, Fuzzy Atom) *Let a domain Ω be given. A fuzzy concept vocabulary over Ω is a triple $(\mathfrak{A}(\Omega), \mu_\cdot, \mathcal{A}_\cdot)$, where $\mathfrak{A}(\Omega)$ is a countable set of symbols and $\mu_\cdot, \mathcal{A}_\cdot$ are mappings*

$$\begin{aligned} \mu_\cdot &: \mathfrak{A}(\Omega) \to F(\Omega), \mathcal{A} \mapsto \mu_\mathcal{A} \\ \mathcal{A}_\cdot &: F(\Omega) \to \mathfrak{A}(\Omega), \mu \mapsto \mathcal{A}_\mu \end{aligned}$$

satisfying $\mu_\cdot \circ \mathcal{A}_\cdot = \mathrm{id}_{F(\Omega)}$.

The elements of $\mathfrak{A}(\Omega)$ are called fuzzy atomic concepts (or fuzzy atoms for short) over Ω. The fuzzy set $\mu_\mathcal{A}$ is called the interpretation of $\mathcal{A}$, and the atom $\mathcal{A}_\mu$ is called the (distinguished) name of μ.

Fuzzy atoms are somewhat similar to linguistic variables in that they decouple an imperfect information from its fuzzy set representation. The mapping $\mathcal{A}_\cdot$ is a technical requirement of our model, as we want to be able to name every fuzzy set within a system. Fuzzy atoms can be defined in a system lexicon that matches vague statements (e.g., "tall", "economical", "bendable") to fuzzy sets over a certain domain, but they can also stem from computations by fuzzy heuristics or provided by imprecise sensors.

A *fuzzy literal* is either an atom or the negation of an atom:

Definition 2.3 (Fuzzy Literal) *A fuzzy literal $\mathcal{L}$ over domain Ω is either a fuzzy atom $\mathcal{A} \in \mathfrak{A}(\Omega)$ or an expression of the form $\overline{\mathcal{A}}$ with $\mathcal{A} \in \mathfrak{A}(\Omega)$. The fuzzy atom $\overline{\mathcal{A}}$ is called the negation of $\mathcal{A}$, its interpretation $\mu_{\overline{\mathcal{A}}}$ is defined as*

$$\mu_{\overline{\mathcal{A}}}(\omega) = 1 - \mu_\mathcal{A}(\omega) \quad \forall \mathcal{A} \in \mathfrak{A}(\Omega), \omega \in \Omega$$

The set of all fuzzy literals over domain Ω is denoted by $\mathfrak{L}(\Omega)$.

Fuzzy literals can be joined together to form *fuzzy clauses,* which are interpreted as the logical disjunction of their constituent literals:

Definition 2.4 (Fuzzy Clause) *A fuzzy clause $\mathcal{K}$ over domain Ω is a finite set of fuzzy literals $\mathcal{L}_i \in \mathfrak{L}(\Omega)$, $1 \le i \le n$.*

We use the notation $\mathcal{K} = \mathcal{L}_1 \vee \mathcal{L}_2 \vee \ldots \vee \mathcal{L}_n$ if we want to emphasize the disjunctive nature of clauses.

The interpretation of a fuzzy clause $\mathcal{K}$ as a fuzzy set $\mu_\mathcal{K}$ is given by

$$\mu_\mathcal{K} \stackrel{\text{def}}{=} \begin{cases} \mu_\perp & \textit{if } \mathcal{K} = \emptyset \\ \mu_{\mathcal{L}_1} \cup \mu_{\mathcal{L}_2} \cup \ldots \cup \mu_{\mathcal{L}_n} & \textit{otherwise} \end{cases}$$

with $\mu_\perp(\omega) := 0 \quad \forall \omega \in \Omega$. The set of all fuzzy clauses over a domain Ω is denoted by $\mathfrak{K}(\Omega)$.

By combining fuzzy clauses with logical *and*-operations we get *fuzzy formulas*:

Definition 2.5 (Fuzzy Formula) *A fuzzy formula $\mathcal{F}$ over domain Ω is a finite set of fuzzy clauses $\mathcal{K}_i \in \mathfrak{K}(\Omega), 1 \le i \le m$. We use the notation $\mathcal{F} = \mathcal{K}_1 \wedge \ldots \wedge \mathcal{K}_m$ if we want to emphasize the conjunctive nature of formulas.*

The interpretation of a fuzzy formula $\mathcal{F}$ as a fuzzy set $\mu_\mathcal{F}$ is given by:

$$\mu_\mathcal{F} \stackrel{\text{def}}{=} \begin{cases} \mu_\top & \textit{if } \mathcal{F} = \emptyset \\ \mu_{\mathcal{K}_1} \cap \mu_{\mathcal{K}_2} \cap \ldots \cap \mu_{\mathcal{K}_m} & \textit{otherwise} \end{cases}$$

with $\mu_\top(\omega) := 1 \quad \forall \omega \in \Omega$. We denote the set of all fuzzy formulas over Ω with $\mathfrak{F}(\Omega)$.

We also call a fuzzy formula a *fuzzy conjunctive normal form* (FCNF).

Example (fuzzy formula) We illustrate the use of fuzzy formulas with a small example from a *fuzzy decision support system* for selecting a car based upon a number of vague requirements.

For this example we assume that a user demands the following features from a car:

not(expensive) ∧ (economical ∨ powerful)

Fuzzy atoms for these vague requirements with their corresponding fuzzy set interpretations might look like this:

	Mercedes	**Porsche**	**VW**	**Smart**
expensive	0.75	1.00	0.50	0.25
economical	0.50	0.25	0.75	1.00
powerful	0.75	1.00	0.50	0.00

If we use these definitions, the above requirement can be modeled as a fuzzy formula $\mathcal{F} = \{\mathcal{K}_1, \mathcal{K}_2\}$ with

$\mathcal{K}_1 = \{\mathcal{L}_1\}$, $\mathcal{K}_2 = \{\mathcal{L}_2, \mathcal{L}_3\}$, $\mathcal{L}_1 = \{\overline{\mathcal{A}_{\text{expensive}}}\}$, $\mathcal{L}_2 = \{\mathcal{A}_{\text{economical}}\}$, and $\mathcal{L}_3 = \{\mathcal{A}_{\text{powerful}}\}$. This fuzzy formula has a fuzzy set interpretation of {*Mercedes/0.25, Porsche/0.0, VW/0.5, Smart/0.75*}, i.e., a *Smart* would be a good (albeit not perfect) candidate for the stated requirements.

The language of fuzzy literals, fuzzy clauses, and fuzzy formulas permits the aggregation of arbitrary pieces of imperfect information, however inconsistent they may be. This is an important feature, because the system does not have to reject inconsistent information out of hand (as is the case with conventional information systems), but can rather attempt to restore consistency by giving preference to some pieces over others. This will be discussed later on in more detail. For now, we just define a notion of consistence:

Definition 2.6 (Degree of Consistency) *The degree of consistency C of a fuzzy clause $\mathcal{K} \in \mathfrak{K}(\Omega)$ is defined as:*

$$C(\mathcal{K}) \stackrel{\text{def}}{=} \sup_{\omega\in\Omega}\{\mu_{\mathcal{K}}(\omega)\}.$$

Analogously, we define the degree of consistency for a fuzzy formula $\mathcal{F} \in \mathfrak{F}(\Omega)$:

$$C(\mathcal{F}) \stackrel{\text{def}}{=} \sup_{\omega\in\Omega}\{\mu_{\mathcal{F}}(\omega)\}.$$

For a pair of fuzzy formulas $\mathcal{F}_1, \mathcal{F}_2$, we define the mutual consistency of $\mathcal{F}_1$ and $\mathcal{F}_2$ as $C(\mathcal{F}_1, \mathcal{F}_2) \stackrel{\text{def}}{=} C(\mathcal{F}_1 \cup \mathcal{F}_2)$ and say that $\mathcal{F}_1$ and $\mathcal{F}_2$ are γ-consistent for $\gamma \in [0, 1]$, if $C(\mathcal{F}_1, \mathcal{F}_2) \geq \gamma$.

We will also need the *negation* of a fuzzy formula:

Definition 2.7 (Negation of Fuzzy Formulas) *The negation $\overline{\mathcal{F}}$ of a fuzzy formula $\mathcal{F} \in \mathfrak{F}(\Omega)$ is computed by pushing the negation to its constituent atoms (using de Morgan's laws) and re-computing the conjunctive normal form.*

However, this definition can lead to an exponential operator execution time. But we can provide a simpler negation for the implementation: for some applications it is sufficient to create a negated fuzzy formula from the negation of its fuzzy interpretation, instead of its logical structure, which can be computed very efficiently.

3 FUZZY BELIEF REVISION

We now address one of the most important points within our representation model: *modifications*. Most existing data models for fuzzy object-oriented systems are only concerned with structural fuzziness, introduced at various points within an object-oriented model (e.g., attributes, objects, inheritance). Information, however, is not static. It is a typical feature of information systems that data changes over time. Thus, a fuzzy object-oriented data model must address modifications of imperfect data. However, we found that there are two typical problems that arise when processing fuzzy information:

- Computations are application specific. The challenge here is to define operators that are abstract enough to be used within a multitude of object-oriented fuzzy information systems, yet specific enough to be still useful.
- Combining different pieces of imperfect information can easily lead to meaningless results. Especially in the case of inconsistent information, fuzzy operations often lead to useless results like "everything is possible" or "nothing is possible".

Since we introduced a new representation model with our fuzzy formulas, we now have to define suitable operators for modifying them. As fuzzy clauses and formulas are sets, modifications could be done by simply adding or removing fuzzy literals or clauses. Simple set operators, however, are not concerned with semantics — clauses added to a fuzzy formula in this way can easily lead to inconsistencies in the fuzzy set interpretation. But maintaining consistency is one of the prime functions of database and information systems.

As mentioned above, we aim to support applications that require the management of complex imperfect states. Examples of systems we want to model are: a fuzzy decision support system that manages vague requirements, a fuzzy reverse engineering framework that maintains information about system artifacts, and a fuzzy noun phrase coreference resolution system that computes and combines results from fuzzy heuristics. Modifications within these applications are not simple updates (in the sense of Katsuno and Mendelzon [KM91]), but rather revisions, where we collect more and more information about an existing or emerging world.

Consequently, our solution is to define fuzzy operators based on Gärdenfors-style belief revision [AGM85, Gär88, Gär92]. They are called γ-expansion, γ-revision, γ-contraction, and allow for modifying a complex fuzzy formula while maintaining a specified degree of consistency γ (in this paper, we only consider γ-expansion and γ-revision). In the case of γ-revision for example, a new piece of information is either rejected, merged with all existing information, or added while removing some fuzzy clauses that are inconsistent to the new fuzzy formula within a specified degree γ.

These operators have the following semantics:

Expansion: A γ-*expansion* adds new information (expressed as a fuzzy formula) to an existing piece of information (again a fuzzy formula), without removing any existing information while ensuring that the resulting fuzzy formula reaches a consistency degree of at least γ. If this is not possible, the new information is *rejected*.

Revision: The γ-*revision* operation always adds new information, if its consistency degree reaches at least γ. Existing information (fuzzy clauses) may be removed in case there is a partial or complete inconsistency.

Contraction: The γ-*contraction* operation removes information from a knowledge set, in a way that preserves as much of the existing information as possible while ensuring that the result's consistency degree reaches at least γ.

In the fuzzy decision support system example, we collect requirements in form of fuzzy formulas. Adding a new requirement (FCNF) always results in a potential conflict — for example, while it is quite understandable that someone wishes for an impressive, big, and rather new car with lots of horsepower, adding a requirement that it must also be very inexpensive will most likely result in a conflict. Traditional fuzzy models would just return a fuzzy set that is close or equal to zero for all cars in the domain. This is rather useless, since it does neither show *why* there is a conflict nor *which* pieces of information actually cause the conflict. Our fuzzy γ-revision, however, can add a new requirement while maintaining a consistent state by showing exactly which existing requirements participate in the conflict and removing some of them.

The following sections show how the AGM postulates can be extended for our fuzzy model. Since we do not consider inference, we formulate the postulates and operators for individual formulas rather then deductively closed sets. This makes our fuzzy belief revision comparable to the base revision [Neb91, Neb92] known from propositional logic. One reason for not considering inference is that it simply is too costly for an (object-oriented) database and information system environment, where efficient implementations of operators are paramount. Another reason is that we are very concerned with the semantic *consistency* of information, which is defined on a formula's model. We therefore examined how syntactic revision can be performed in absence of an inference mechanism by directly checking the semantic interpretation of a fuzzy formula through the consistency operator C defined above. This consistency operator can be implemented very efficiently on the fuzzy set interpretations. We therefore based the notion of consistency on the interpretation of a fuzzy formula when we translated the AGM postulates for our model, and subsequently examined the features of operators satisfying these translated postulates.

3.1 FUZZY EXPANSION

Expansion is a monotonous extension of a fuzzy formula $\mathcal{F}$ by a fuzzy formula $\mathcal{G}$, i.e., none of the fuzzy clauses in $\mathcal{F}$ are removed. More precisely, we have:

Definition 3.1 (γ-Expansion) *A γ-expansion operator $+_\gamma$ is a mapping that takes two fuzzy formulas $\mathcal{F}, \mathcal{G} \in \mathfrak{F}(\Omega)$ and a value $\gamma \in [0, 1]$ to a new fuzzy formula in $\mathfrak{F}(\Omega)$:*

$$+_\gamma : \mathfrak{F}(\Omega) \times \mathfrak{F}(\Omega) \to \mathfrak{F}(\Omega)$$

and satisfies the expansion postulates $E_\gamma 1$–$E_\gamma 6$ (see below).

The semantics of the γ-expansion operator is defined by the following postulates, here $\mathcal{F}, \mathcal{G}$ denote fuzzy formulas from $\mathfrak{F}(\Omega)$:

E_γ1 (Stability) The result of a γ-expansion is always a fuzzy formula:

$$\mathcal{F} +_\gamma \mathcal{G} \in \mathfrak{F}(\Omega).$$

E_γ2a (Success) If the consistency degree of $\mathcal{F} \cup \mathcal{G}$ is at least γ, the γ-expansion is successful:

$$\text{If } C(\mathcal{F} \cup \mathcal{G}) \geq \gamma, \text{then } \mathcal{G} \subseteq \mathcal{F} +_\gamma \mathcal{G}.$$

E_γ2b (Failure) If the consistency degree of $\mathcal{F} \cup \mathcal{G}$ is less than γ, the original formula remains unchanged:

$$\text{If } C(\mathcal{F} \cup \mathcal{G}) < \gamma, \text{then } \mathcal{F} +_\gamma \mathcal{G} = \mathcal{F}.$$

E_γ3 (Expansion) The set of information expands:

$$\mathcal{F} +_\gamma \mathcal{G} \supseteq \mathcal{F}.$$

E_γ4 (Invariance) If the new information $\mathcal{G}$ is already known, invariance holds:

$$\mathcal{G} \subseteq \mathcal{F} \Rightarrow \mathcal{F} +_\gamma \mathcal{G} = \mathcal{F}.$$

E_γ5 (Monotonicity) Let $\mathcal{F}' \subseteq \mathcal{F}$. If the expansion $\mathcal{F} +_\gamma \mathcal{G}$ is successful or the expansion $\mathcal{F}' +_\gamma \mathcal{G}$ fails, monotonicity holds:

$$\text{If } (\mathcal{G} \subseteq \mathcal{F} +_\gamma \mathcal{G}) \vee (\mathcal{G} \not\subseteq \mathcal{F}' +_\gamma \mathcal{G}),$$
$$\text{then } \mathcal{F}' +_\gamma \mathcal{G} \subseteq \mathcal{F} +_\gamma \mathcal{G}.$$

E_γ6 (Predictability) The result of a γ-expansion must only contain fuzzy clauses from $\mathcal{F}$ or $\mathcal{G}$:

$$\mathcal{F} +_\gamma \mathcal{G} \subseteq \mathcal{F} \cup \mathcal{G}.$$

It is easy to see that these postulates admit only a single expansion operator, namely the one defined by

$$\mathcal{F} +_\gamma \mathcal{G} \stackrel{\text{def}}{=} \begin{cases} \mathcal{F} \cup \mathcal{G} & \text{if } C(\mathcal{F} \cup \mathcal{G}) \geq \gamma, \\ \mathcal{F} & \text{otherwise.} \end{cases}$$

Thus, in case of fuzzy formulas, γ-expansion is actually a quite simple operation: the result is the union of the clause sets of both fuzzy formulas if the union reaches the consistency degree γ. Otherwise, the result is the original formula $\mathcal{F}$, i.e., the new information $\mathcal{G}$ is rejected.

The main difference between this operation and the classical expansion is that we do not always add the new information. Since our main goal is to guarantee consistency within an information system, we reject new information if it is not consistent to the existing information within a specified degree γ.

Example (Fuzzy Expansion) We continue the fuzzy decision support example. Within this context, the fuzzy expansion operator can be used to implement a simple decision support algorithm that adds new requirements while making sure they are compatible with all existing requirements. Suppose a user started with the requirement not(*expensive*), represented by a fuzzy formula $\mathcal{F}_1 = \{\mathcal{K}_1\}$, with $\mathcal{K}_1 = \{\overline{\mathcal{A}_{\text{expensive}}}\}$. He now demands that the car should also be *economical*, represented by a fuzzy clause $\mathcal{K}_2 = \{\mathcal{A}_{\text{economical}}\}$. Adding this request through γ-expansion with a consistency degree of 0.75:

$$\mathcal{F}_2 = \mathcal{F}_1 +_{0.75} \{\mathcal{K}_2\} = \{\mathcal{K}_1, \mathcal{K}_2\}$$

leads to a new fuzzy formula $\mathcal{F}_2$, i.e., the expansion is successful, and its fuzzy set interpretation $\mu_{\mathcal{F}_2}$ is {*Mercedes/0.25, Porsche/0.0, VW/0.5, Smart/0.75* }. So, a good candidate for the stated requirements would be a *Smart*. Impressed by this result, the user requests that the car should also be *spacious*, represented by a fuzzy atom $\mathcal{A}_{\text{spacious}}$ that has the following fuzzy set interpretation:

	Mercedes	Porsche	VW	Smart
spacious	1.0	0.25	0.75	0.0

However, the γ-expansion operator rejects this request, i.e., the operation $\mathcal{F}_2 +_{0.75} \{\mathcal{K}_3\}$ results in an unchanged $\mathcal{F}_2$, since adding the new requirement would lead to a fuzzy set interpretation that has a consistency degree of only 0.5 < γ = 0.75.

3.2 FUZZY REVISION

So far we rejected all inconsistent information to guarantee the integrity of our knowledge base. But this not a generally feasible approach, since most applications cannot turn away all new information, just because they do not fit into the established world. In these cases it becomes necessary to remove some existing information to preserve a consistent state. Because of this requirement, γ-revision is a *non-monotonous* operation. However, it is not practical to simply remove all existing information, which would be a trivial way to always ensure a consistent belief state. What makes belief revision a hard problem is that there is usually no unique result that incorporates all new information while preserving as much existing information as possible. Since our model is designed to work with information systems, efficiency becomes a prime objective (most of the more theoretical oriented research in classical belief revision would result in very inefficient implementations).

We begin again by defining a formal notion of γ-revision and then show the possible revision operators:

Definition 3.2 (γ-Revision) *A γ-revision operator $\oplus_\gamma$ is a mapping that takes two fuzzy formulas $\mathcal{F}, \mathcal{G} \in \mathfrak{F}(\Omega)$ and a value $\gamma \in [0, 1]$ to a fuzzy formula from $\mathfrak{F}(\Omega)$:*

$$\oplus_\gamma : \mathfrak{F}(\Omega) \times \mathfrak{F}(\Omega) \rightarrow \mathfrak{F}(\Omega)$$

and satisfies the revision postulates $R_\gamma 1$–$R_\gamma 6$ (see below).

We now present our translation of the original revision postulates by Gärdenfors. To make it easier to compare our postulates with the original, we kept the original numbering, adding letter indices in cases where we splitted a single postulate.

Here, $\mathcal{F}, \mathcal{G}, \mathcal{H}$ are fuzzy formulas over $\mathfrak{F}(\Omega)$:

R_γ1 (Stability) The result of a revision operation is always a fuzzy formula:

$$\mathcal{F} \oplus_\gamma \mathcal{G} \in \mathfrak{F}(\Omega).$$

R_γ2a (Priority) If the new fuzzy formula $\mathcal{G}$ has a consistency degree of at least γ, then it must be contained in the result:

$$\text{If } C(\mathcal{G}) \geq \gamma, \text{ then } \mathcal{G} \subseteq \mathcal{F} \oplus_\gamma \mathcal{G}.$$

R_γ2b (Failure) If the consistency degree of the new fuzzy formula is less than γ, the original formula remains unchanged:

$$\text{If } C(\mathcal{G}) < \gamma, \text{ then } \mathcal{F} \oplus_\gamma \mathcal{G} = \mathcal{F}.$$

R_γ3 (Predictability) The result of a revision contains only fuzzy clauses from $\mathcal{F}$ or $\mathcal{G}$:

$$\mathcal{F} \oplus_\gamma \mathcal{G} \subseteq \mathcal{F} \cup \mathcal{G}.$$

R$_\gamma$4 (Compatibility) If the fuzzy clauses of the new formula are compatible with the existing clauses, the expansion is a subset of the revision:

$$\text{If } \mathcal{G} \subseteq \mathcal{F} +_\gamma \mathcal{G} \text{ and } C(\mathcal{F} +_\gamma \mathcal{G}) \geq \gamma, \\ \text{then } \mathcal{F} \oplus_\gamma \mathcal{G} \supseteq \mathcal{F} +_\gamma \mathcal{G}.$$

R$_\gamma$5 (Consistency) In case of a successful revision, the consistency degree must reach at least γ:

$$\text{If } C(\mathcal{G}) \geq \gamma, \text{ then } C(\mathcal{F} \oplus_\gamma \mathcal{G}) \geq \gamma.$$

R$_\gamma$6 (Identity) The interpretation of the result does not depend on the syntactic form of $\mathcal{G}$:

$$\text{If } \mu_\mathcal{G} = \mu_\mathcal{H}, \text{ then } \mu_{\mathcal{F} \oplus_\gamma \mathcal{G}} = \mu_{\mathcal{F} \oplus_\gamma \mathcal{H}}.$$

These are the so-called *basic postulates* that must be satisfied by every revision operator. There are two supplementary postulates that control iterative changes:

R$_\gamma$7 If two fuzzy formulas $\mathcal{G}$ and $\mathcal{H}$ can be successfully added to a formula $\mathcal{F}$ by revision and expansion, the result is a superset of the revision of $\mathcal{F}$ with $\mathcal{G} \cup \mathcal{H}$:

$$\text{If } \mathcal{G} \cup \mathcal{H} \subseteq (\mathcal{F} \oplus_\gamma \mathcal{G}) +_\gamma \mathcal{H}, \text{ then} \\ \mathcal{F} \oplus_\gamma (\mathcal{G} \cup \mathcal{H}) \subseteq (\mathcal{F} \oplus_\gamma \mathcal{G}) +_\gamma \mathcal{H}.$$

R$_\gamma$8 Extension of postulate R$_\gamma$4:

$$\text{If } \mathcal{G} \cup \mathcal{H} \subseteq (\mathcal{F} \oplus_\gamma \mathcal{G}) +_\gamma \mathcal{H}, \text{ then} \\ (\mathcal{F} \oplus_\gamma \mathcal{G}) +_\gamma \mathcal{H} \subseteq \mathcal{F} \oplus_\gamma (\mathcal{G} \cup \mathcal{H}).$$

The semantics of most postulates is straightforward. Noteworthy is postulate R$_\gamma$6, which ensures that the interpretation of the result of a γ-revision is independent of the syntax of the involved fuzzy formulas: if two fuzzy formulas have a different syntax but the same interpretation, the results of the revisions with these two formulas must also have the same interpretation.

There are a number of additional properties that follow from the postulates: firstly, if the γ-expansion $\mathcal{F} +_\gamma \mathcal{G}$ is successful, the result of the γ-revision $\mathcal{F} \oplus_\gamma \mathcal{G}$ is equal to the expansion. Secondly, if the new information is already known ($\mathcal{G} \subseteq \mathcal{F}$) and satisfies the consistency requirement ($C(\mathcal{F}) \geq \gamma$), then $\mathcal{F}$ remains unchanged: $(\mathcal{G} \subseteq \mathcal{F}) \wedge (C(\mathcal{F}) \geq \gamma) \Rightarrow \mathcal{F} \oplus_\gamma \mathcal{G} = \mathcal{F}$, i.e., the γ-revision is invariant with respect to known information. And thirdly, if an operation satisfies both supplementary postulates, the sets $(\mathcal{F} \oplus_\gamma \mathcal{G}) +_\gamma \mathcal{H}$ and $\mathcal{F} \oplus_\gamma (\mathcal{G} \cup \mathcal{H})$ are actually equal: If $\mathcal{G} \subseteq \mathcal{F} \oplus_\gamma \mathcal{G}$ and $\mathcal{H} \subseteq (\mathcal{F} \oplus_\gamma \mathcal{G}) +_\gamma \mathcal{H}$, then $(\mathcal{F} \oplus_\gamma \mathcal{G}) +_\gamma \mathcal{H} = \mathcal{F} \oplus_\gamma (\mathcal{G} \cup \mathcal{H})$. We omit the proofs for these claims since they are quite straightforward.

There are two additional properties, which we will need for the following revision theorem:

Lemma 3.3 *Let $\mathcal{F} \in \mathfrak{F}(\Omega)$ and $\mu \in F(\Omega) - \{\mu_\perp\}$ arbitrary. Then a fuzzy formula $\mathcal{G} \in \mathfrak{F}(\Omega)$ exists with $\mathcal{F} \cap \mathcal{G} = \emptyset$ and $\mu_\mathcal{G} = \mu$.*

Proof. *It is easy to see that there are infinitely many fuzzy clauses $\mathcal{K} \in \mathfrak{K}(\Omega)$ with $\mu_\mathcal{K} = \mu$. Only a finite number of these can be contained in $\mathcal{F}$. Therefore, select a clause $\mathcal{K} \in \mathfrak{K}(\Omega) - \mathcal{F}$ with $\mu_\mathcal{K} = \mu$ and let $\mathcal{G} := \{\mathcal{K}\}$.* ■

Lemma 3.4 *Let $\mathcal{F}, \mathcal{G}_1, \mathcal{G}_2 \in \mathfrak{F}(\Omega)$ and $\gamma \in [0,1]$ arbitrary. If $\mu_{\mathcal{G}_1} = \mu_{\mathcal{G}_2}$ and $C(\mathcal{G}_1) = C(\mathcal{G}_2) \geq \gamma$, then*

$$(\mathcal{F} \oplus_\gamma \mathcal{G}_1) \cup \mathcal{G}_2 = (\mathcal{F} \oplus_\gamma \mathcal{G}_2) \cup \mathcal{G}_1$$

If additionally $\mathcal{F}, \mathcal{G}_1, \mathcal{G}_2$ are pairwise disjunct, it holds that

$$(\mathcal{F} \oplus_\gamma \mathcal{G}_1) - \mathcal{G}_1 = (\mathcal{F} \oplus_\gamma \mathcal{G}_2) - \mathcal{G}_2$$

Proof. *Since $\mu_{\mathcal{G}_1} = \mu_{\mathcal{G}_2}$ it holds that $C((\mathcal{F} \oplus_\gamma \mathcal{G}_1) \cup \mathcal{G}_2) = C((\mathcal{F} \oplus_\gamma \mathcal{G}_1) \cup \mathcal{G}_1) = C(\mathcal{F} \oplus_\gamma \mathcal{G}_1) \geq \gamma$, and analogously $C((\mathcal{F} \oplus_\gamma \mathcal{G}_2) \cup \mathcal{G}_1) \geq \gamma$. From R$_\gamma$7 and R$_\gamma$8 it follows that*

$$\begin{aligned} (\mathcal{F} \oplus_\gamma \mathcal{G}_1) \cup \mathcal{G}_2 &= \mathcal{F} \oplus_\gamma (\mathcal{G}_1 \cup \mathcal{G}_2) \\ &= (\mathcal{F} \oplus_\gamma \mathcal{G}_2) \cup \mathcal{G}_1 \end{aligned}$$

and therefore the proposition. ■

The next step is to obtain a more precise description of the possible revision operators satisfying R$_\gamma$1–R$_\gamma$8. It turns out that the following simple characterization can be given:

Theorem 3.5 (Revision theorem) *Let $\mathcal{F} \in \mathfrak{F}(\Omega)$ and $\gamma \in [0,1]$ be arbitrary, but fixed. A mapping $\mathcal{F}^{\oplus_\gamma} : \mathfrak{F}(\Omega) \to \mathfrak{F}(\Omega)$, $\mathcal{G} \mapsto \mathcal{F} \oplus_\gamma \mathcal{G}$ satisfies the postulates R$_\gamma$1–R$_\gamma$8 if and only if an enumeration $2^\mathcal{F} = \{\mathcal{F}_0, \mathcal{F}_1, \ldots, \mathcal{F}_{2^{|\mathcal{F}|}-1}\}$ of the power set of $\mathcal{F}$ exists with*

$$\mathcal{F} \oplus_\gamma \mathcal{G} = \begin{cases} \mathcal{F} & \text{if } C(\mathcal{G}) < \gamma, \\ \mathcal{F} \cup \mathcal{G} & \text{if } C(\mathcal{F} \cup \mathcal{G}) \geq \gamma, \\ \mathcal{F}_i \cup \mathcal{G} & \text{otherwise} \end{cases}$$

where $i = \min\{j | C(\mathcal{G} \cup \mathcal{F}_j) \geq \gamma\}$.

Proof. *The direction "⇐" (i.e., such an operator satisfies postulates R$_\gamma$1–R$_\gamma$8) is easy to see.*

Therefore, we only show the direction "⇒". The cases $C(\mathcal{G}) < \gamma$ and $C(\mathcal{F} \cup \mathcal{G}) \geq \gamma$ are trivial. The interesting case is $C(\mathcal{G}) \geq \gamma \wedge C(\mathcal{F} \cup \mathcal{G}) < \gamma$. Let $\oplus_\gamma$ be a fixed operator satisfying R$_\gamma$1–R$_\gamma$8. Define an enumeration $\mathcal{F}_0, \mathcal{F}_1, \ldots, \mathcal{F}_{2^{|\mathcal{F}|}-1}$ of the sub-formulas of $\mathcal{F}$ by induction:

$$\begin{aligned} \mathcal{F}_0 &:= \mathcal{F} \\ \mathcal{F}_{i+1} &:= \begin{cases} (\mathcal{F} \oplus_\gamma \mathcal{G}_i) - \mathcal{G}_i & \text{if } C(\mathcal{G}_i) \geq \gamma \\ \textit{else arbitrary from } 2^\mathcal{F} - \{\mathcal{F}_0, \ldots, \mathcal{F}_i\} \end{cases} \end{aligned}$$

whereas $\mathcal{G}_i \in \mathfrak{F}(\Omega)$ is an arbitrary formula with $\mathcal{F} \cap \mathcal{G}_i = \emptyset$ and

$$\mu_{\mathcal{G}_i}(\omega) = \begin{cases} 1 & \text{if } \mu_{\mathcal{F}_j}(\omega) < \gamma \text{ for } 0 \le j \le i, \\ 0 & \text{else} \end{cases}$$

The existence of such $\mathcal{G}_i$ and the independence of the $\mathcal{F}_i$ from the selected $\mathcal{G}_i$ follows from the Lemmas 3.3 and 3.4.

Now let $\mathcal{G} \in \mathfrak{F}(\Omega)$ with $C(\mathcal{G}) \ge \gamma$, $C(\mathcal{F} \cup \mathcal{G}) < \gamma$ be arbitrary selected. Then the set $\{j | C(\mathcal{G} \cup \mathcal{F}_j) \ge \gamma\}$ is not empty (it contains at least the j with $\mathcal{F}_j = \emptyset$), and it does not contain $\mathcal{F}_0 = \mathcal{F}$. Therefore, $i := \min\{j | C(\mathcal{G} \cup \mathcal{F}_j) \ge \gamma\}$ is well-defined, and it holds that $i > 0$. We have to show that $\mathcal{F} \oplus_\gamma \mathcal{G} = \mathcal{G} \cup \mathcal{F}_i$.

Let $\varepsilon > 0$ arbitrary. From $R_\gamma 5$ follows $C(\mathcal{F} \oplus_\gamma \mathcal{G}) \ge \gamma$, hence there exists an $\omega \in \Omega$ with $\mu_{\mathcal{F} \oplus_\gamma \mathcal{G}}(\omega) \ge \gamma - \varepsilon$. From $R_\gamma 2a$ then follows $\mu_{\mathcal{G}}(\omega) \ge \gamma - \varepsilon$. Moreover, it follows from the selection of i that for $0 \le j < i$ holds: $C(\mathcal{G} \cup \mathcal{F}_j) < \gamma$, in particular $\mu_{\mathcal{F}_j}(\omega) < \gamma$. For this reason $\mu_{\mathcal{G}_{i-1}}(\omega) = 1$ and consequently $C((\mathcal{F} \oplus_\gamma \mathcal{G}) \cup \mathcal{G}_{i-1}) \ge \min(\mu_{\mathcal{F} \oplus_\gamma \mathcal{G}}(\omega), \mu_{\mathcal{G}_{i-1}}(\omega)) \ge \gamma - \varepsilon$. Since $\varepsilon > 0$ is arbitrary, $C((\mathcal{F} \oplus_\gamma \mathcal{G}) \cup \mathcal{G}_{i-1}) \ge \gamma$ also holds. From $R_\gamma 7$ and $R_\gamma 8$ then follows

$$(\mathcal{F} \oplus_\gamma \mathcal{G}) \cup \mathcal{G}_{i-1} = \mathcal{F} \oplus_\gamma (\mathcal{G} \cup \mathcal{G}_{i-1})$$

Furthermore, because of $C(\mathcal{G} \cup \mathcal{F}_i) \ge \gamma$ there exists an $\omega' \in \Omega$ with $\mu_{\mathcal{G}}(\omega') \ge \gamma - \varepsilon$ and $\mu_{\mathcal{F}_i}(\omega') \ge \gamma - \varepsilon$. Like above it follows $\mu_{\mathcal{G}_{i-1}}(\omega') = 1$ and $C(\mathcal{F}_i \cup \mathcal{G}_{i-1} \cup \mathcal{G}) \ge \gamma$. By definition of $\mathcal{F}_i$ it holds that $\mathcal{F}_i \cup \mathcal{G}_{i-1} = ((\mathcal{F} \oplus_\gamma \mathcal{G}_{i-1}) - \mathcal{G}_{i-1}) \cup \mathcal{G}_{i-1} = (\mathcal{F} \oplus_\gamma \mathcal{G}_{i-1}) \cup \mathcal{G}_{i-1} = \mathcal{F} \oplus_\gamma \mathcal{G}_{i-1}$, whereby the last equality follows from $C(\mathcal{G}_{i-1}) \ge \mu_{\mathcal{G}_{i-1}}(\omega') = 1$ and $R_\gamma 2a$. Thus $C((\mathcal{F} \oplus_\gamma \mathcal{G}_{i-1}) \cup \mathcal{G}) = C(\mathcal{F}_i \cup \mathcal{G}_{i-1} \cup \mathcal{G}) \ge \gamma$. From $R_\gamma 7$ and $R_\gamma 8$ then follows:

$$(\mathcal{F} \oplus_\gamma \mathcal{G}_{i-1}) \cup \mathcal{G} = \mathcal{F} \oplus_\gamma (\mathcal{G} \cup \mathcal{G}_{i-1})$$

and in combination with the corresponding equation above

$$(\mathcal{F} \oplus_\gamma \mathcal{G}_{i-1}) \cup \mathcal{G} = (\mathcal{F} \oplus_\gamma \mathcal{G}) \cup \mathcal{G}_{i-1}$$

Since $\mathcal{G}_{i-1}$ can be selected disjunctive from $\mathcal{F} \cup \mathcal{G}$ without loss of generality, this can be transformed to

$$((\mathcal{F} \oplus_\gamma \mathcal{G}_{i-1}) - \mathcal{G}_{i-1}) \cup \mathcal{G} = \mathcal{F} \oplus_\gamma \mathcal{G}$$

and by inserting the definition of $\mathcal{F}_i$ we receive the proposition. ■

Obviously, only a subset of all possible operators maximize the result set. We therefore introduce an additional postulate:

$R_\gamma 9$ (Preservation) A revision operator must compute the maximal possible result set:

$$C(\mathcal{G}) \ge \gamma \wedge \mathcal{H} \subseteq \mathcal{F} \cup \mathcal{G} \wedge \mathcal{H} \supsetneq \mathcal{F} \oplus_\gamma \mathcal{G} \Rightarrow C(\mathcal{H}) < \gamma$$

I.e., there must be no clause from $\mathcal{F} \cup \mathcal{G}$ that can be added to the result of a revision without falling short of the requested consistency degree. It is easy to see that a revision operator satisfies $R_\gamma 9$ if and only if for every fixed formula $\mathcal{F}$ the enumeration of $2^{\mathcal{F}}$ given by Theorem 3.5 is compatible with the superset of $2^{\mathcal{F}}$, i.e., $\mathcal{F}_i \supseteq \mathcal{F}_j \Rightarrow i < j$.

3.2.1 Revision Based on Orders

According to Theorem 3.5, every revision operator is obtained by providing an ordering of the subsets of $\mathcal{F}$, for every fuzzy formula $\mathcal{F}$. In practice, it is often more natural to provide an ordering of the elements of $\mathcal{F}$, i.e., the clauses constituting $\mathcal{F}$. For example, if a γ-revision with $\mathcal{G}$ forces some clauses of $\mathcal{F}$ to be eliminated, but there is a choice as to which ones, it might be preferable to retain clauses that have more recently been added to $\mathcal{F}$, or those that have a higher degree of internal consistency. We call such preference orders among clauses *epistemic relevance orders*, in analogy to [Neb90].

More precisely, we say that an epistemic relevance order for a fuzzy formula $\mathcal{F}$ is a total ordering $\le$ of the clauses of $\mathcal{F}$. If an epistemic relevance order $\le$ is given for the clauses of a formula $\mathcal{F}$, an ordering of the subsets of $\mathcal{F}$ can be constructed by extending $\le$ in the natural way:

$$\begin{aligned} \mathcal{F}_1 < \mathcal{F}_2 \quad &\stackrel{\text{def}}{\Longleftrightarrow} \quad (\exists \mathcal{K} : \mathcal{K} \in \mathcal{F}_2 \wedge \mathcal{K} \notin \mathcal{F}_1 \\ &\qquad \wedge (\forall \mathcal{K}' \ge \mathcal{K} : \\ &\qquad\quad (\mathcal{K}' \in \mathcal{F}_1 \Leftrightarrow \mathcal{K}' \in \mathcal{F}_2))), \\ \mathcal{F}_1 \le \mathcal{F}_2 \quad &\stackrel{\text{def}}{\Longleftrightarrow} \quad \mathcal{F}_1 < \mathcal{F}_2 \vee \mathcal{F}_1 = \mathcal{F}_2, \end{aligned}$$

where $\mathcal{F}_1$ and $\mathcal{F}_2$ are subsets of $\mathcal{F}$. In other words, a subset $\mathcal{F}_2$ of $\mathcal{F}$ is more relevant than a subset $\mathcal{F}_1$ if and only if when examining the clauses of $\mathcal{F}$ in order of decreasing relevance, the first clause distinguishing $\mathcal{F}_1$ and $\mathcal{F}_2$ belongs to $\mathcal{F}_2$.

This ordering of the subsets of $\mathcal{F}$, in turn, can be used to define a revision operator by enumerating the subsets of $\mathcal{F}$ in order of decreasing relevance and then applying Theorem 3.5. More precisely, we have the following:

Definition 3.6 (γ-Revision with Epistemic Relevance) *Let $\mathcal{F}, \mathcal{G} \in \mathfrak{F}(\Omega)$ be fuzzy formulas, $\le$ an epistemic relevance order for $\mathcal{F}$ and $\gamma \in [0, 1]$ arbitrary. Let $\mathcal{K}_0, \ldots, \mathcal{K}_{n-1}$ be the clauses of $\mathcal{F}$ numbered in ascending relevance order and let $\mathcal{F}_0, \mathcal{F}_1, \ldots, \mathcal{F}_{2^{|n|}-1}$ be the enumeration of the subsets of $\mathcal{F}$ obtained by assigning to each subset $\{\mathcal{K}_{i_1}, \ldots, \mathcal{K}_{i_n}\}$ of $\mathcal{F}$ the index i whose binary representation has a zero at position $i_1, \ldots, i_n$ and ones elsewhere. Set*

$$\mathcal{F} \oplus_\gamma^\le \mathcal{G} := \begin{cases} \mathcal{F} & \text{if } C(\mathcal{G}) < \gamma, \\ \mathcal{F} \cup \mathcal{G} & \text{if } C(\mathcal{F} \cup \mathcal{G}) \ge \gamma, \\ \mathcal{F}_i \cup \mathcal{G} & \text{else,} \end{cases}$$

where $i = \min\{j | C(\mathcal{F}_j \cup \mathcal{G}) \geq \gamma\}$*. Then the mapping* $(\mathcal{F}, \mathcal{G}) \mapsto \mathcal{F} \oplus_\gamma^\leq \mathcal{G}$ *is called the* γ*-revision operator based on the relevance order* $\leq$*.*

Theorem 3.5 guarantees that the mapping $\oplus_\gamma^\leq$ defined above is indeed a revision operator. Moreover, we have:

Theorem 3.7 (Maximal result set) *Every revision operator based on epistemic relevance satisfies the maximality postulate* $R_\gamma 9$*.*

Proof. *It is easy to see that for every fuzzy formula* $\mathcal{F}$*, the enumeration of* $2^\mathcal{F}$ *defined above is compatible with the superset ordering, i.e. if* $\mathcal{F}_i \supseteq \mathcal{F}_j$*, then* $i \leq j$*.* ■

Although constructive, Definition 3.6 is not directly usable for implementing a revision operator based on epistemic relevance, because searching all subsets of $\mathcal{F}$ for the lowest-numbered one compatible with $\mathcal{G}$ would be unacceptably slow. Fortunately, $\mathcal{F} \oplus_\gamma^\leq \mathcal{G}$ can be computed in a much more direct way:

Theorem 3.8 (Efficient γ-revision operator with epistemic relevance) *Let* $\mathcal{F}, \mathcal{G}, \gamma$ *be as in Definition 3.6. Then*

$$\mathcal{F} \oplus_\gamma^\leq \mathcal{G} = \begin{cases} \mathcal{F} & \text{if } C(\mathcal{G}) < \gamma, \\ ((\mathcal{G} +_\gamma \{\mathcal{K}_{n-1}\}) & \\ \quad +_\gamma \ldots) +_\gamma \{\mathcal{K}_0\} & \text{else} \end{cases}$$

where $\mathcal{K}_0, \ldots, \mathcal{K}_{n-1}$ *are the clauses of* $\mathcal{F}$ *numbered in ascending relevance order.*

Proof. *The claim is obviously true if* $C(\mathcal{G}) < \gamma$*, so suppose that* $C(\mathcal{G}) \geq \gamma$*. Let* $2^\mathcal{F} = \{\mathcal{F}_0, \ldots, \mathcal{F}_{2^n-1}\}$ *be an enumeration of the subsets of* $\mathcal{F}$ *as in Definition 3.6, i.e., if* $\mathcal{F}_j = \{\mathcal{K}_{i_1}, \ldots, \mathcal{K}_{i_n}\}$*, then the binary representation of* j *has zeros in position* $i_1, \ldots, i_n$ *and ones elsewhere. Let* $\mathcal{F}_i$ *and* $\mathcal{F}_j$ *be the largest subsets of* $\mathcal{F}$ *satisfying* $\mathcal{F} \oplus_\gamma^\leq \mathcal{G} = \mathcal{G} \cup \mathcal{F}_i$ *and* $((\mathcal{G} +_\gamma \{\mathcal{K}_{n-1}\}) +_\gamma \ldots) +_\gamma \{\mathcal{K}_0\} = \mathcal{G} \cup \mathcal{F}_j$*, respectively. Since* $C(\mathcal{G} \cup \mathcal{F}_j) = C(\mathcal{G} +_\gamma \ldots) \geq \gamma$ *and* $i = \min\{k | C(\mathcal{G} \cup \mathcal{F}_k) \geq \gamma\}$*,* $i \leq j$ *must hold. Suppose now, for the purpose of reaching a contradiction, that* i *were strictly less than* j*. In this case, there would be a bit position, say* p*, where the binary expansion of* i *had a zero and the binary expansion of* j *a one, with all bits to the left of* p *being equal. In terms of clauses, this would imply* $\mathcal{K}_p \in \mathcal{F}_i, \mathcal{K}_p \notin \mathcal{F}_j$ *and* $\mathcal{K}_q \in \mathcal{F}_i \Leftrightarrow \mathcal{K}_q \in \mathcal{F}_j$ *for* $n > q > p$*. But then*

$$\begin{aligned} &(((\mathcal{G} +_\gamma \{\mathcal{K}_{n-1}\}) +_\gamma \ldots) +_\gamma \{\mathcal{K}_{p+1}\}) \cup \{\mathcal{K}_p\} \\ &= \mathcal{G} \cup \{\mathcal{K}_q | \mathcal{K}_q \in \mathcal{F}_j \wedge n > q > p\} \cup \{\mathcal{K}_p\} \\ &\subseteq \mathcal{G} \cup \mathcal{F}_i \\ &= \mathcal{G} \oplus_\gamma^\leq \mathcal{F} \end{aligned}$$

hence $C((((\mathcal{G} +_\gamma \{\mathcal{K}_{n-1}\}) +_\gamma \ldots) +_\gamma \{\mathcal{K}_{p+1}\}) \cup \{\mathcal{K}_p\}) \geq C(\mathcal{G} \oplus_\gamma^\leq \mathcal{F}) \geq \gamma$*, hence by virtue of* $E_\gamma 2a$ $\mathcal{K}_p \in (((\mathcal{G} +_\gamma \{\mathcal{K}_{n-1}\}) +_\gamma \ldots) +_\gamma \{\mathcal{K}_{p+1}\}) +_\gamma \mathcal{K}_p \subseteq \mathcal{G} \cup \mathcal{F}_j$*, and hence (because* $\mathcal{F}_j$ *was chosen to be maximal)*

$$\mathcal{K}_p \in \mathcal{F}_j,$$

thus reaching the desired contradiction. ■

Example (Fuzzy Revision) We continue the car decision example. Remember that the γ-expansion rejected the user's demand for a *spacious* car, since the result would be below the requested consistency degree. The user now has the option to *force* this request through γ-revision, resulting in:

$$\mathcal{F}_3 = \mathcal{F}_2 \oplus_{0.75} \{\mathcal{K}_3\} = \{\mathcal{K}_3, \mathcal{K}_2\},$$

i.e., the clause $\mathcal{K}_1$ representing the requirement not(*expensive*) has been removed in order to maintain a consistent state (we assume the clauses have been ordered, with $\mathcal{K}_1 < \mathcal{K}_2$). The result $\mathcal{F}_3$ has a fuzzy set interpretation of $\mu_{\mathcal{F}_3} = \{$ *Mercedes/0.5, Porsche/0.25, VW/0.75, Smart/0.0* $\}$, i.e., a *Volkswagen* would be a good fit for the revised requirements *economical* $\wedge$ *spacious*.

4 CONCLUSIONS

Fuzzy information systems have new requirements for both modeling and manipulating imperfect data, which cannot adequately be satisfied with existing approaches. In this paper, we defined a fuzzy representation model that combines both syntax and semantics, which allows for a strict separation of the structure of imperfect information, modeled as fuzzy atoms, fuzzy clauses, and fuzzy formulas, and their semantic interpretation in form of fuzzy sets. Modifications on these fuzzy formulas can be performed by the operators γ-expansion and γ-revision. Both always guarantee a specified consistency degree — an important feature for database and information system practitioners.

Since these operations abstract from the manipulation of single fuzzy sets, more complex application-specific algorithms dealing with imperfection can be developed in a more natural way.

The model presented here has been successfully incorporated into an object-oriented data model, which in turn has been implemented with a number of class libraries for the *Java* language. An accompanying fuzzy information system architecture provides the necessary extensions to a classical architecture for creating fuzzy information systems. With this approach we have been able to selectively enhance information systems with the capabilities to represent complex imperfect information and process them in a robust and consistency-preserving way. So far, we successfully built a fuzzy decision support system and a fuzzy noun phrase coreference system. Both proved to be easier to design and implement than comparative classical, non-fuzzy systems while at the same time providing new features like adjustable certainty degrees.

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